Topic 3

VECTORS

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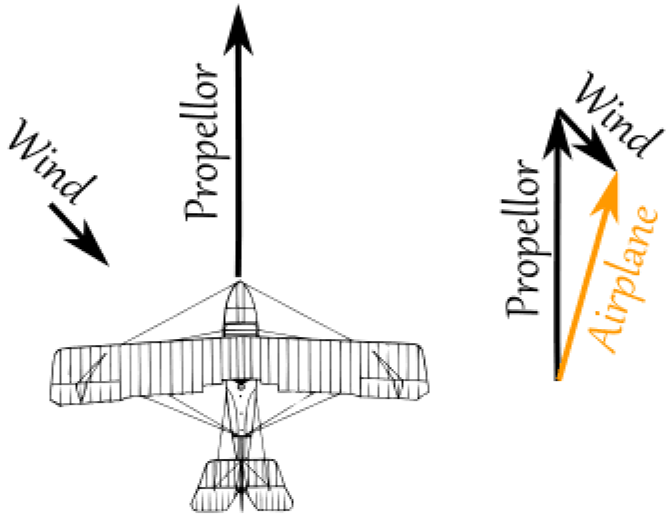

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3.1 VECTORS IN TWO AND THREE DIMENSIONS

3.1.1 Geometric Vectors

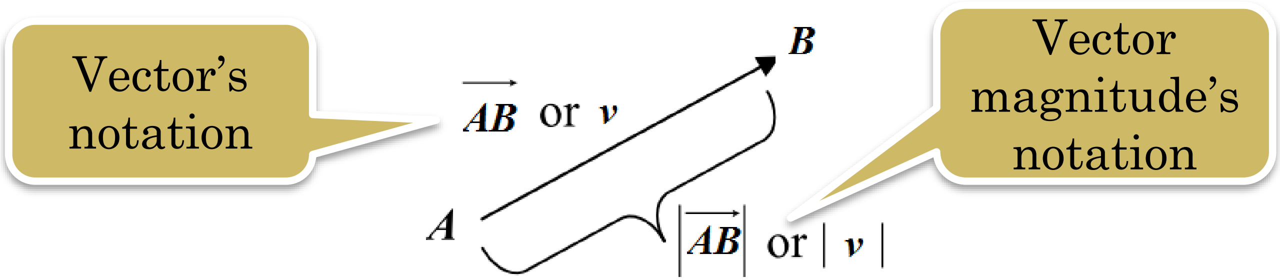
Physical Quantities

Scalar	Vector
Has magnitude only	Has magnitude and direction
For example: speed, length, volume, mass, energy and temperature are scalars	For example: velocity, acceleration, force and momentum are vectors.

3.1.1 Geometric Vectors

Vectors are usually represented geometrically as **directed line** segments or arrows.

The tail of the arrow is called the **initial point** of the vector, and the tip of the arrow the **terminal point**.



The initial point of a vector **v** is A and the terminal point is B, we write $\mathbf{v} = \overrightarrow{AB}$.

3.1.1 Geometric Vectors

Two vectors are regarded as **equal** (or equivalent) if both vectors are of the **same direction** with **equal magnitude,** regardless of the position of the initial points.

The vector of length zero is called the **zero vector** or **null vector** and is denoted by $\mathbf{0} = \langle 0,0 \rangle$ whose magnitude is zero and whose direction is indeterminate.

3.1 VECTORS IN TWO AND THREE DIMENSIONS

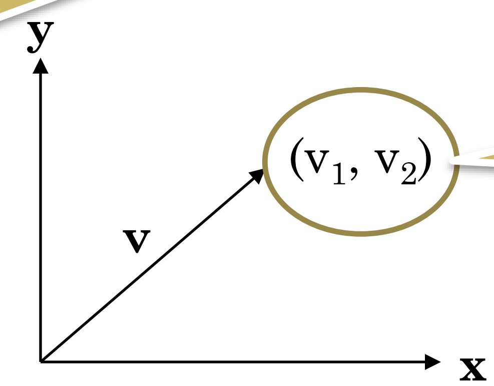
3.1.2 Vectors in a 2-dimension

If **v** is a vector in the plane whose initial point is the origin and terminal point is (v_1, v_2), then the coordinates (v_1, v_2) of the terminal point of **v** are called the **component** of **v** and we write

$$\mathbf{v} = \langle v_1, v_2 \rangle$$

A **vector** denoted by angle bracket

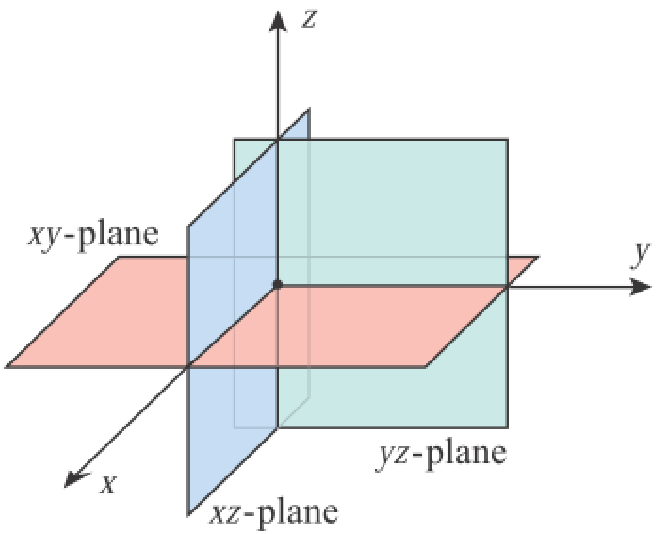
v_1 and v_2 are **component**



A **point** is denoted by parenthesis

3.1.3 Vectors in a 3-dimension

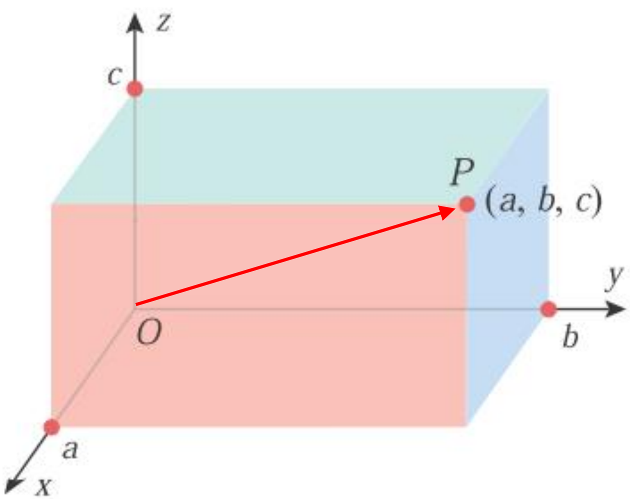
There are three planes: *xy-plane*, *xz*-plane and *yz*-plane. These three planes form eight octants; the **first octant** consists of all positive numbers (+x, +y,+z). The remaining seven octants have different combinations of positive and negative numbers: (-x, +y, +z), (-x, -y, +z), (+x, -y, +z), (+x, +y, -z), (-x, +y, -z), (-x, -y, -z) and (+x, -y, +z).



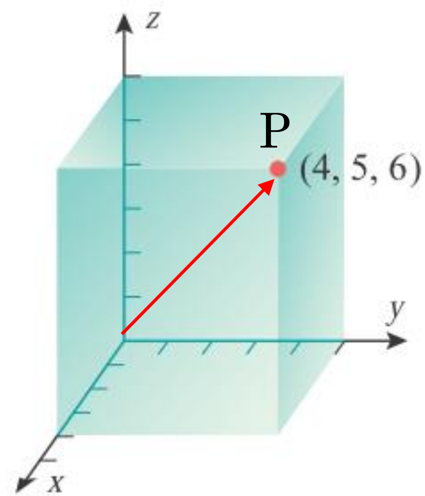
REGION	DESCRIPTION
xy-plane	Consists of all points of the form (*x*, *y*, 0)
xz-plane	Consists of all points of the form (*x*, 0, *z*)
yz-plane	Consists of all points of the form (0, *y*, *z*)
x-axis	Consists of all points of the form (*x*, 0, 0)
y-axis	Consists of all points of the form (0, *y*, 0)
z-axis	Consists of all points of the form (0, 0, *z*)

3.1 VECTORS IN TWO AND THREE DIMENSIONS

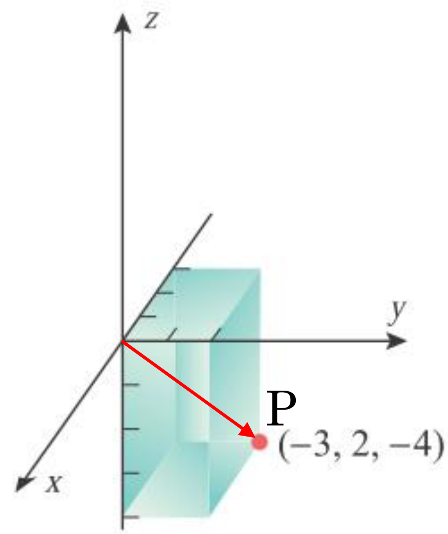
3.1.3 Vectors in a 3-dimension



P = <a, b, c>



P = <4, 5, 6>



P = <-3, 2, -4>

3.1 VECTORS IN TWO AND THREE DIMENSIONS

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are points in 2D space, then the vector joining initial point A to terminal point B, denoted by $\overrightarrow{AB}$, has the **component form**

$$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1 \rangle.$$

Similarly, if $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are points in 3D space, then

$$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

For the special case when initial point A is the **origin**, i.e. either $O(0,0)$ or $O(0,0,0)$, then the resulting vector $\overrightarrow{OB}$ is a special type of vector known as a **position vector**.

3.1 VECTORS IN TWO AND THREE DIMENSIONS

Example:

In 2-space the vector from $P_1(1, 3)$ to $P_2(4, -2)$ is

Solution:

$$\overrightarrow{P_1P_2} = \langle 4 - 1, -2 - 3 \rangle = \langle 3, -5 \rangle$$

Example:

In 3-space the vector from $A(0, -2, 5)$ to $B(3, 4, -1)$ is

Solution:

$$\overrightarrow{AB} = \langle 3 - 0, 4 - (-2), -1 - 5 \rangle = \langle 3, 6, -6 \rangle$$

3.1.4 Properties of Vector

1. If $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ are equivalent, then $v_1 = w_1$, $v_2 = w_2$ and $v_3 = w_3$.

2. $\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$.

3. $k\mathbf{v} = \langle kv_1, kv_2, kv_3 \rangle$ where k is any scalar.

4. If the vector $\overrightarrow{P_1P_2}$ has initial point $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ terminal point, then the component form of the vector **v** represented by $\overrightarrow{P_1P_2}$ is

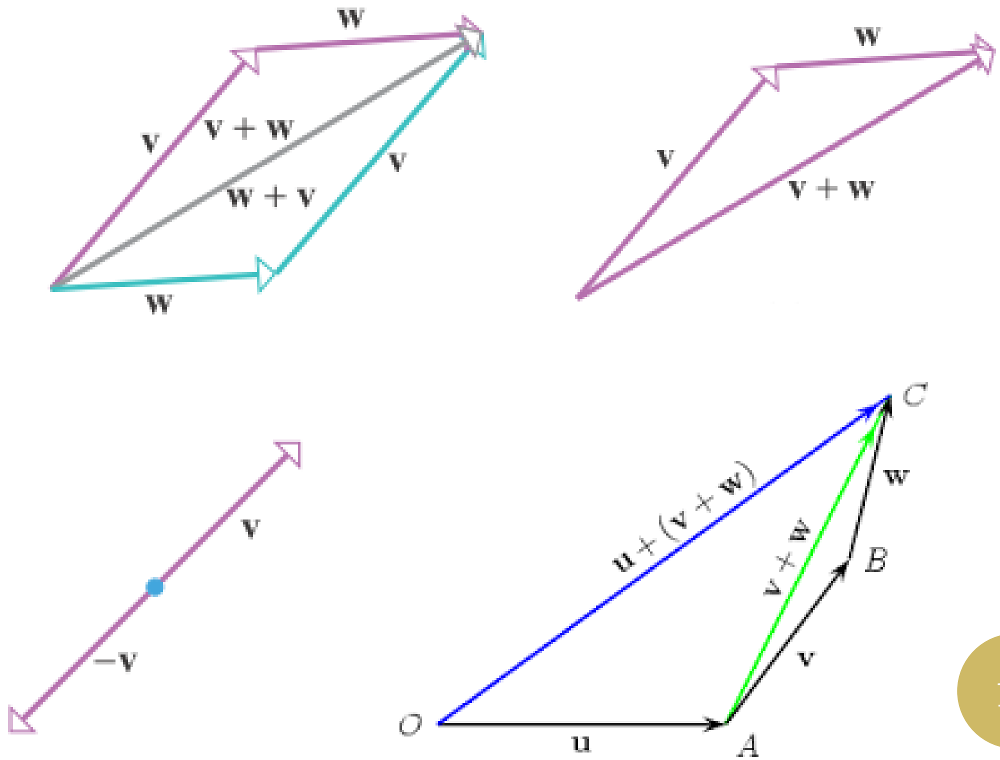
   $$\langle v_1, v_2, v_3 \rangle = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

3.1 VECTORS IN TWO AND THREE DIMENSIONS

3.1.5 Properties of Vector Operations

If **u**, **v**, and **w** are vectors in 2 or 3-space and *k* and *l* are scalars, the following relationships hold.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $k(l\mathbf{u}) = (kl)\mathbf{u}$
6. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
7. $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$
8. $1\mathbf{u} = \mathbf{u}$



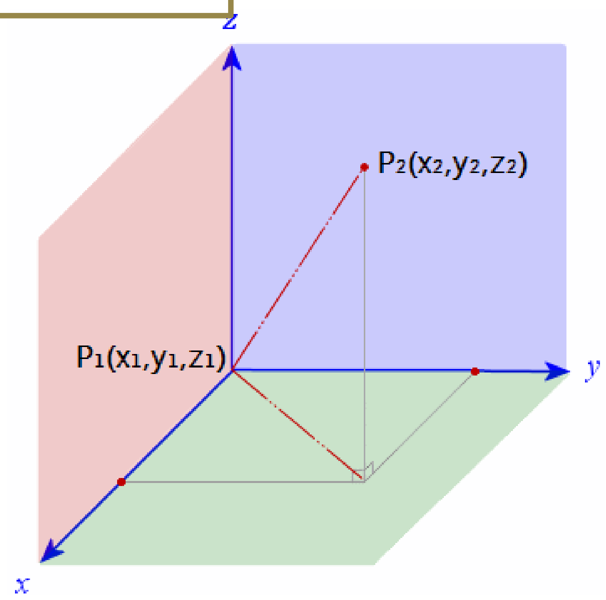
3.1.6 Norm of a Vector

The **length** of a vector **v** is often called the **norm** of **v** and is denoted by $|\mathbf{v}|$ where

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2} \quad \text{in 2-space}$$

and

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} \quad \text{in 3-space}$$



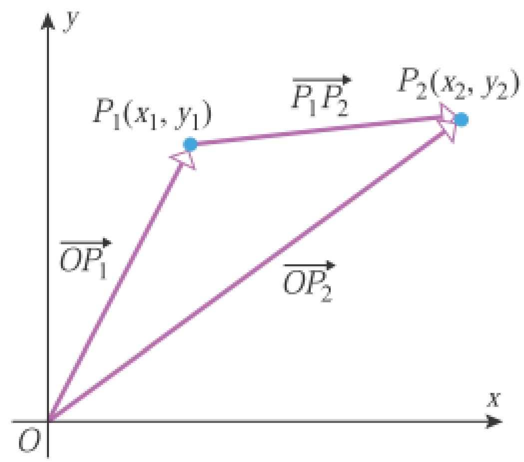
3.1 VECTORS IN TWO AND THREE DIMENSIONS

3.1.6 Norm of a Vector

Distance Formula in Two Dimensions with initial point not at origin

The distance between the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

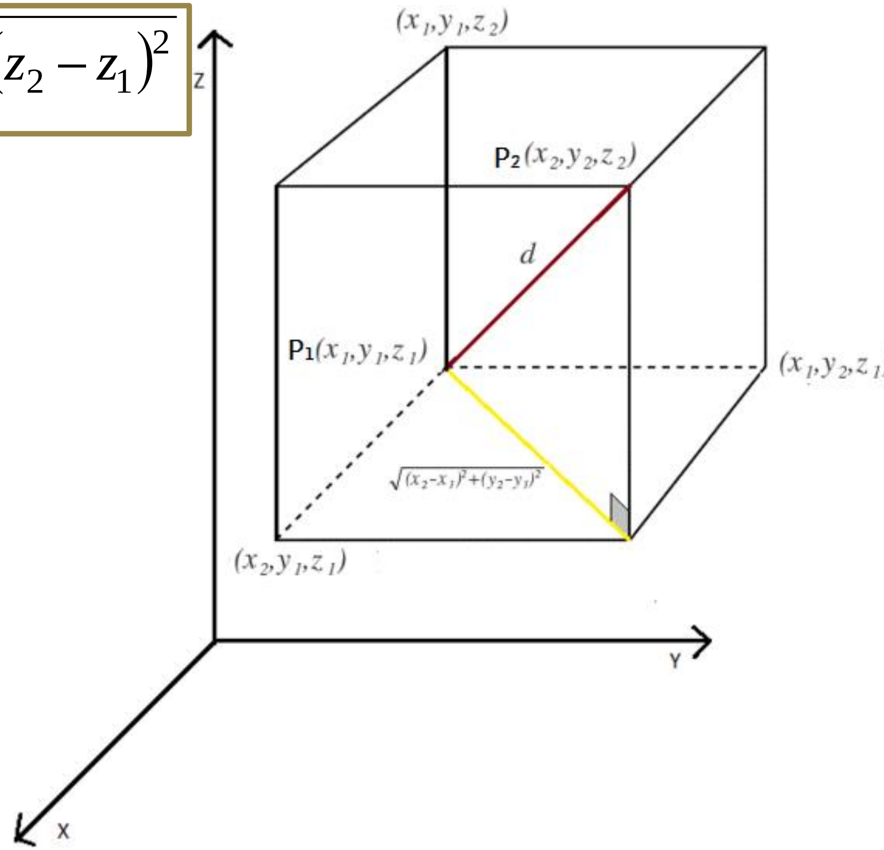


3.1.6 Norm of a Vector

Distance Formula in Three Dimensions with initial point not at origin

The distance between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



3.1 VECTORS IN TWO AND THREE DIMENSIONS

Example:
Given points $P_1(2,-1,-5)$ and $P_2(4,-3,1)$, find the norm of the vector **v** represented by $\overrightarrow{P_1P_2}$

Solution:

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(4-2)^2 + (-3+1)^2 + (1+5)^2} = \sqrt{44} = 2\sqrt{11}$$

3.1 VECTORS IN TWO AND THREE DIMENSIONS

Example:

Given $\mathbf{u} = < 3, -2 >$ *and* $\mathbf{v} = \langle -9, 0, 2 \rangle$, *find* $|\mathbf{u}|$ *and* $|\mathbf{v}|$.

Solution:

$|\mathbf{u}| = \sqrt{3^2 + (-2)^2} = \sqrt{9+4} = \sqrt{13}$.

$|\mathbf{v}| = \sqrt{(-9)^2 + 0^2 + (2)^2} = \sqrt{81+4} = \sqrt{85}$.

3.2 UNIT VECTORS

3.2.1 Unit Vector in the Direction of v

If **v** is a nonzero vector in the plane, then the unit vector **u** is defined as

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{|\mathbf{v}|}\mathbf{v}$$

and has magnitude (or length) 1 and the same direction as **v**.

3.2 UNIT VECTORS

Example:
Find a unit vector in the direction of $\mathbf{v} = \langle 1,1,2 \rangle$ and verify that the result has length 1.
Solution:
The unit vector in the direction of **v** is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle 1,1,2 \rangle}{\sqrt{1^2+1^2+2^2}} = \frac{1}{\sqrt{6}}\langle 1,1,2 \rangle = \left\langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle$$

The vector has length 1, because

$$\sqrt{\left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{2}{\sqrt{6}}\right)^2} = \sqrt{\frac{1}{6}+\frac{1}{6}+\frac{4}{6}} = 1$$

3.2 UNIT VECTORS

Example:

Determine whether the following vectors are unit vectors.

i) $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$; *ii)* $\mathbf{v} = \langle 2, 1, 0 \rangle$.

Solution:

i) $|\mathbf{u}| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$. $\therefore \mathbf{u}$ is a unit vector.

ii) $|\mathbf{v}| = \sqrt{2^2 + 1^2 + 0^2} = \sqrt{4+1} = \sqrt{5}$. $\therefore \mathbf{v}$ is not a unit vector.

3.2 UNIT VECTORS

3.2.2 Standard Unit Vector

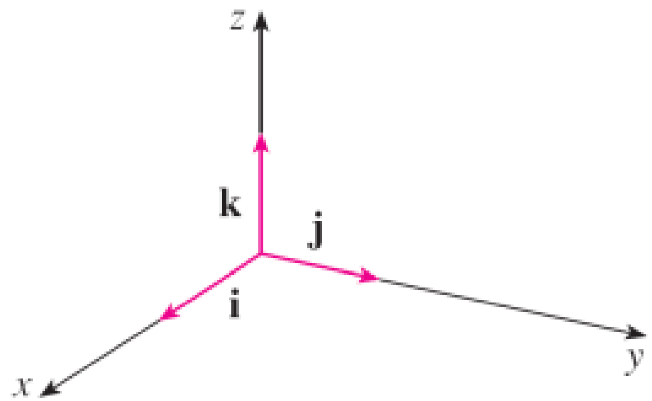
The unit vectors $\mathbf{i} = \langle 1,0,0 \rangle$, $\mathbf{j} = \langle 0,1,0 \rangle$ and $\mathbf{k} = \langle 0,0,1 \rangle$ are called the **standard unit vectors** in the plane and we can write

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1 \langle 1,0,0 \rangle + v_2 \langle 0,1,0 \rangle + v_3 \langle 0,0,1 \rangle = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$$

For example, $\langle 3,-2,1 \rangle = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

in components form

in standard basis vectors (**i**, **j** and **k** terms)



3.3 Dot Product/ Scalar Product

3.3.1 Component Form of the Dot Product

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be two vectors in 2-space, then their **dot product** are as follows:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$$

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be two vectors in 3-space, then their **dot product** are as follows:

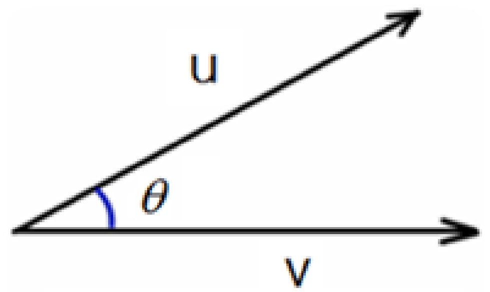
$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

3.3 Dot Product/ Scalar Product

3.3.1 Component Form of the Dot Product

If θ (where $0 \le \theta \le \pi$) is the angle between **u** and **v**, then the **dot product** can also be defined as

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta$$

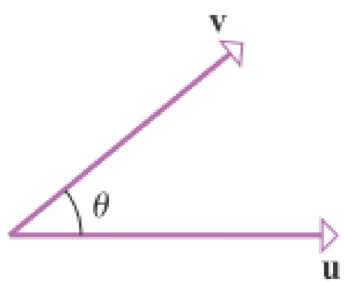


$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta$$

3.3 Dot Product/ Scalar Product

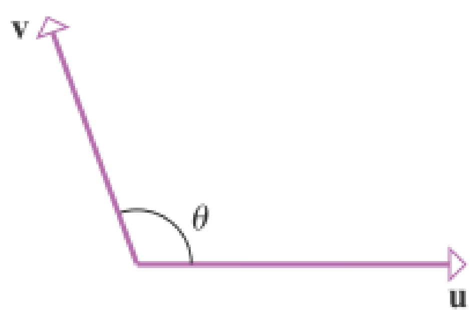
3.3.1 Component Form of the Dot Product

Interpreting the sign of the dot product $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$



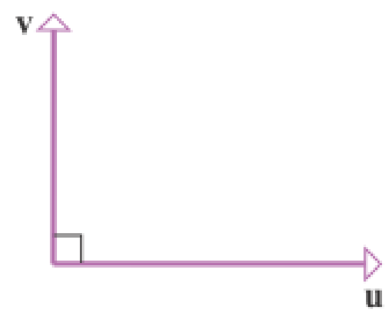
$\mathbf{u} \cdot \mathbf{v} > 0$

the two vectors point in the **same** direction



$\mathbf{u} \cdot \mathbf{v} < 0$

the two vectors point in the **opposite** direction



$\mathbf{u} \cdot \mathbf{v} = 0$

if two vectors are **orthogonal/ perpendicular/ normal**

3.3 Dot Product

Example:
Consider the vector $\mathbf{u} = \langle 2, -1, 1 \rangle$ and $\mathbf{v} = \langle 1, 1, 2 \rangle$, find their dot product and determine the angle between **u** and **v.**

Solution:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = (2)(1) + (-1)(1) + (1)(2) = 3$$

$$|\mathbf{u}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$$

$$|\mathbf{v}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$$

$$\Rightarrow \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \frac{3}{\sqrt{6}\sqrt{6}} = \frac{1}{2}$$

$$\therefore \theta = 60^\circ$$

3.3 Dot Product

Example:
Find the dot product of $\mathbf{u} = \langle 0,0,1 \rangle$ and $\mathbf{v} = \langle 0,2,2 \rangle$ where $\theta = 45^\circ$

Solution:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta = \left(\sqrt{0^2 + 0^2 + 1^2}\right)\left(\sqrt{0^2 + 2^2 + 2^2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{4} = 2$$

3.3 Dot Product

Example:
Show that $\mathbf{u} = \langle 2, 2, -1 \rangle$ and $\mathbf{v} = \langle 5, -4, 2 \rangle$ are perpendicular

Solution:
If **u** and **v** are perpendicular, then $\mathbf{u} \cdot \mathbf{v} = 0$

$$\mathbf{u} \cdot \mathbf{v} = 2(5) + 2(-4) + (-1)(2) = 0$$

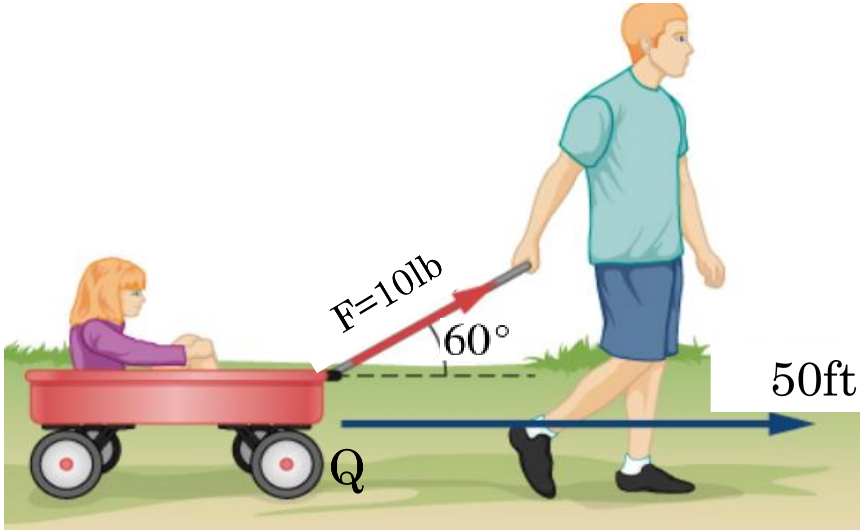
3.3 Dot Product

A wagon is pulled horizontally by exerting a constant force of 10 lb on the handle at an angle of 60° with the horizontal. How much work is done in moving the wagon 50 ft?

Solution:

With $\|\mathbf{F}\| = 10$, $\theta = 60°$, and $\|\overrightarrow{PQ}\| = 50$, it follows that the work done is

$$W = (\|\mathbf{F}\| \cos\theta)\|\overrightarrow{PQ}\| = 10 \cdot \frac{1}{2} \cdot 50 = 250 \text{ ft·lb}$$



3.3 Dot Product

3.3.2 Properties of the Dot Product

Properties of the Dot Product

If **u**, **v** and **w** are vectors in 2- or 3-space and k is a scalar, then

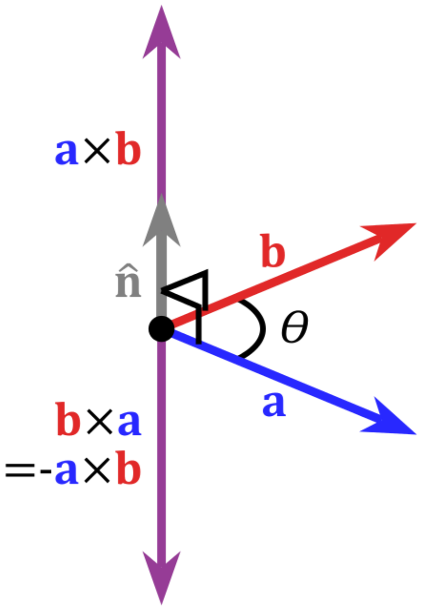
1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
3. $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$
4. $\mathbf{v} \cdot \mathbf{v} > 0$ if $\mathbf{v} \neq \mathbf{0}$, and $\mathbf{v} \cdot \mathbf{v} = 0$ if $\mathbf{v} = \mathbf{0}$
5. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
6. Two nonzero vectors **u** and **v** are **orthogonal/perpendicular** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$, we write $\mathbf{u} \perp \mathbf{v}$.

3.4 Cross Product/ Vector Product

Some of the concept that we will develop in this section requires basic ideas about determinants.

A **2 x 2** determinant is $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$

A **3 x 3** determinant is $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$



3.4 Cross Product

3.4.1 Definition of Cross Product

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be two vectors in 3-space, then their **cross product** **u x v** is the vector defined by:

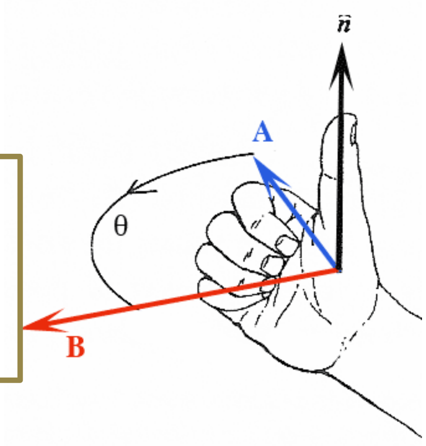
$$\mathbf{u} \times \mathbf{v} = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle$$

or in the determinant notation,

$$\mathbf{u} \times \mathbf{v} = \left\langle \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) \right\rangle$$

The cross product can be calculated as follows:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$



3.4 Cross Product

Example:
If $\mathbf{u} = \langle 1,2,-2 \rangle$ and $\mathbf{v} = \langle 3,0,1 \rangle$, find the cross product of **u** and **v**.

Solution:

$$\mathbf{u} \text{ x } \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \mathbf{k} = 2\mathbf{i} - 7\mathbf{j} - 6\mathbf{k}$$

3.4 Cross Product

3.4.2 Properties of Cross Product

If **u**, **v** and **w** are any vectors in 3-space and *k* is a scalar, then:

1. **u** x **v** = - (**v** x **u**)
2. **u** x (**v** + **w**) = (**u** x **v**) + (**u** x **w**)
3. (**u** + **v**) x **w** = (**u** x **w**) + (**v** x **w**)
4. *k*(**u** x **v**) = *k*(**u**) x **v** = **u** x (*k***v**) = (**u** x **v**) *k*
5. **u** x **0** = **0** x **u** = **0**
6. **u x u = 0**
7. **u** and **v** are parallel if and only if **u x v= 0**

Remark:

Cross product is neither commutative nor associative, that is,

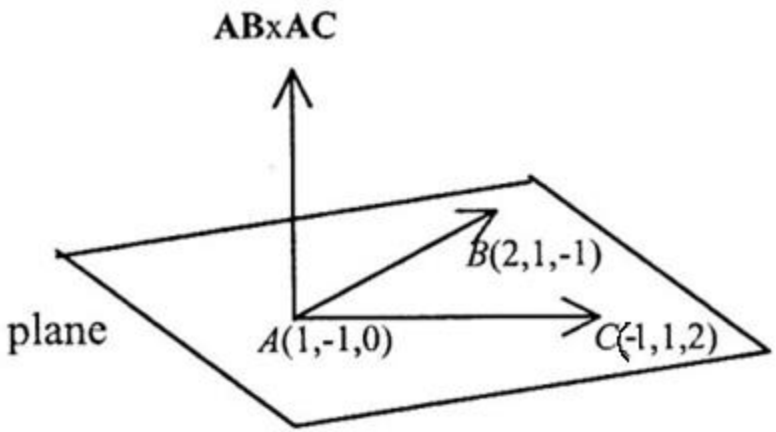
$$\mathbf{a}\times\mathbf{b}\neq\mathbf{b}\times\mathbf{a},$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\neq(\mathbf{a}\times\mathbf{b})\times\mathbf{c}.$$

3.4 Cross Product

Example:

Find a vector perpendicular to the plane that contains $A(1,-1,0)$, $B(2,1,-1)$ *and* $C(-1,1,2)$.



3.4 Cross Product

Solution:

We begin by forming the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ which lie on the plane, i.e.

$$\overrightarrow{AB} = (2-1)\hat{\mathbf{i}} + (1+1)\hat{\mathbf{j}} + (-1-0)\hat{\mathbf{k}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$$

$$\overrightarrow{AC} = (-1-1)\hat{\mathbf{i}} + (1+1)\hat{\mathbf{j}} + (2-0)\hat{\mathbf{k}} = -2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}.$$

Taking their cross product yields

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix}\hat{\mathbf{i}} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix}\hat{\mathbf{j}} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}\hat{\mathbf{k}}$$

$$= 6\hat{\mathbf{i}} + 6\hat{\mathbf{k}}.$$

The vector $6\hat{\mathbf{i}} + 6\hat{\mathbf{k}}$ is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$, which means that it is also perpendicular to the plane on which they lie.

3.4 CROSS PRODUCT

Example:

Show that $\mathbf{a} = \langle 2,9,3 \rangle$ *and* $\mathbf{b} = \left\langle -1, -\tfrac{9}{2}, -\tfrac{3}{2} \right\rangle$ *are parallel.*

Solution:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 9 & 3 \\ -1 & -\frac{9}{2} & -\frac{3}{2} \end{vmatrix}$$

$$= \left[(9)\left(-\tfrac{3}{2}\right) - \left(-\tfrac{9}{2}\right)(3)\right]\hat{\mathbf{i}} - \left[2\left(-\tfrac{3}{2}\right) - (-1)(3)\right]\hat{\mathbf{j}} + \left[2\left(-\tfrac{9}{2}\right) - (-1)(9)\right]\hat{\mathbf{k}}$$

$$= 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}} = \mathbf{0}$$

$\therefore \mathbf{a}$ and $\mathbf{b}$ are parallel.

3.4 Cross Product

3.4.3 Relationship Involving Cross Product and Dot Product

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are any vectors in 3-space , then:

1. $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ($\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$)
2. $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ($\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{v}$)
3. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$
4. $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$

3.4 Cross Product

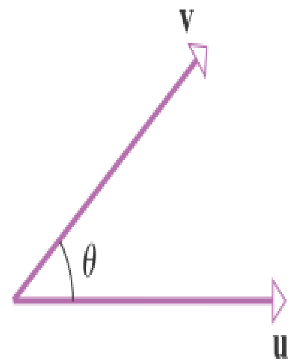
3.4.4 Cross Product and Angle of Vectors

If **u** and **v** are vectors and θ is the angle between **u** and **v**, then

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta \qquad \text{where} \quad 0 \le \theta \le \pi$$

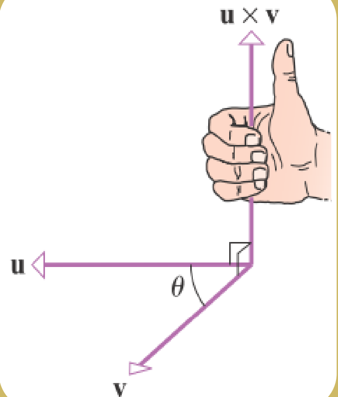
SUMMARY:

Dot Product



- the product of two vectors is a **scalar**
- defined in 2D and 3D space
- $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$.
- $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$.
- if $\mathbf{u} \cdot \mathbf{v} = 0$,then **u** and **v** are **perpendicular**

Cross Product



- the product of two vectors is a **vector**
- defined only in 3D space
- $\mathbf{u} \text{ x } \mathbf{v} = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle$.
- $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$. (area of the parallelogram)
- **If** $\mathbf{u} \times \mathbf{v} = 0$ **,then u** and **v** are **parallel**

3.5 EQUATIONS OF LINES AND PLANES

3.5.1 Lines in 3-Dimensional

If L is a line in 3D-space through the **point** $P_0(x_0, y_0, z_0)$ and **parallel** to the nonzero **vector** $\mathbf{v}=\langle a,b,c\rangle$, then this line L is represented by the **vector equation**

$$\mathbf{r}=\mathbf{r_0}+t\mathbf{v}$$

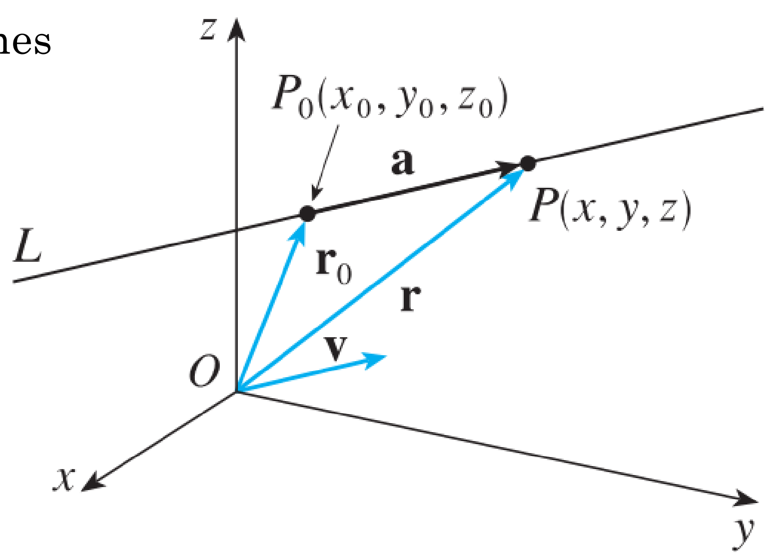
where t is scalar

- $\mathbf{r_0}$: a given point, P_o
- $\mathbf{v}$: direction(parallel) from a given vector, **v**

based on the vector addition using Triangle Law $\mathbf{r}=\mathbf{r_0}+\mathbf{a}$

If $\mathbf{r}=\langle x,y,z\rangle$, the vector equation becomes

$$\langle x,y,z\rangle=\langle x_0+ta, y_0+tb, z_0+tc\rangle$$



3.5.1 Lines in 3-Dimensional

Parametric Equations of a Line in Space

It can be written in scalar equations,

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

which are called **parametric equations**

Symmetric Equations of a Line in Space

If a vector $\mathbf{v} = \langle a, b, c \rangle$ is used to describe the direction of a line *L*, then the numbers *a, b,* and *c* are called **direction numbers** of *L*.

If the direction numbers *a*, *b* and *c* are all nonzero, then we can eliminate the parameter *t* to obtain the **symmetric equations** of a line.

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

3.5 EQUATIONS OF LINES AND PLANES

Example:

Find the parametric equations and symmetric equations of the line that pass through the points (1,2,-3) and parallel to the vector $\mathbf{v} = \langle 4,5,-7 \rangle$

Solution:

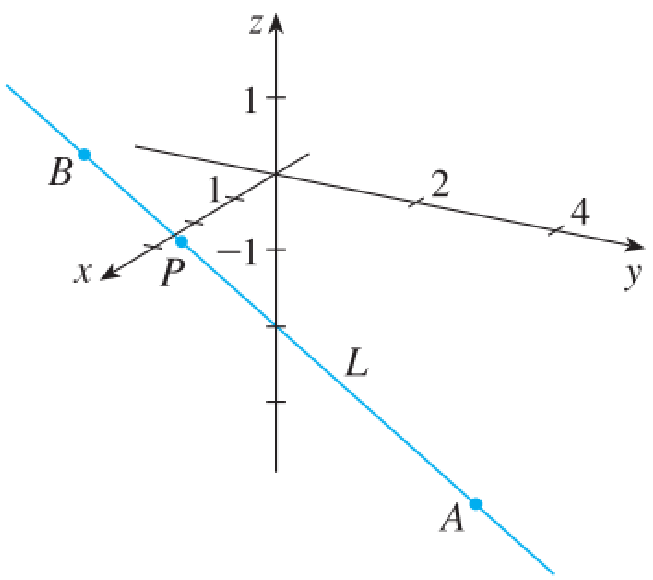
The parametric equations are : $x = 1 + 4t \quad y = 2 + 5t \quad z = -3 - 7t$

The symmetric equations are : $\dfrac{x-1}{4} = \dfrac{y-2}{5} = \dfrac{z+3}{-7}$

3.5 EQUATIONS OF LINES AND PLANES

Example:

(a) Find parametric equations and symmetric equations of the line that passes through the points $A(2, 4, -3)$ and $B(3, -1, 1)$.
(b) At what point does this line intersect the xy-plane?



3.5 EQUATIONS OF LINES AND PLANES

Example:

(a) Find parametric equations and symmetric equations of the line that passes through the points $A(2, 4, -3)$ and $B(3, -1, 1)$.

Solution:

(a) We are not explicitly given a vector parallel to the line, but observe that the vector $\mathbf{v}$ with representation $\overrightarrow{AB}$ is parallel to the line and

$$\mathbf{v} = \langle 3 - 2, -1 - 4, 1 - (-3) \rangle = \langle 1, -5, 4 \rangle$$

Thus direction numbers are $a = 1$, $b = -5$, and $c = 4$. Taking the point $(2, 4, -3)$ as P_0, we see that parametric equations (2) are

$$x = 2 + t \qquad y = 4 - 5t \qquad z = -3 + 4t$$

and symmetric equations (3) are

$$\frac{x - 2}{1} = \frac{y - 4}{-5} = \frac{z + 3}{4}$$

3.5 EQUATIONS OF LINES AND PLANES

Example:

(b) At what point does this line intersect the xy-plane?

Solution:

(b) The line intersects the xy-plane when $z = 0$, so we put $z = 0$ in the symmetric equations and obtain

$$\frac{x-2}{1} = \frac{y-4}{-5} = \frac{3}{4}$$

This gives $x = \frac{11}{4}$ and $y = \frac{1}{4}$, so the line intersects the xy-plane at the point $\left(\frac{11}{4}, \frac{1}{4}, 0\right)$.

3.5 EQUATIONS OF LINES AND PLANES

3.5.2 Planes in 3-Dimensional

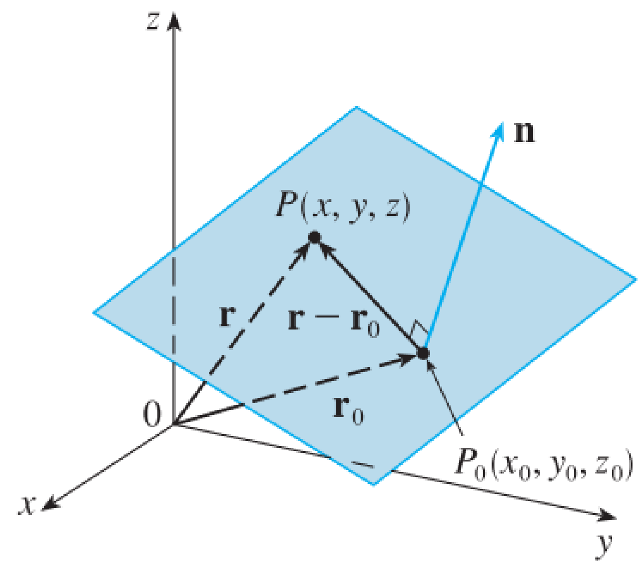
The plane containing the **point** $P_0\,(x_0, y_0, z_0)$ and having a **normal vector** $\mathbf{n} = \langle a, b, c \rangle$ is **perpendicular /orthogonal** to every vector in the given plane, in particular to $\mathbf{r} - \mathbf{r_0}$, so by the dot product of orthogonal vectors, we have

a given point, P_o

normal vector (direction)

$$\mathbf{n}.(\mathbf{r} - \mathbf{r_0}) = 0$$

which can be written as $\mathbf{n}.\mathbf{r} = \mathbf{n}.\mathbf{r_0}$
is called the **vector equation** of the plane.



3.5.2 Planes in 3-Dimensional

By writing $\mathbf{n} = \langle a,b,c \rangle$, $\mathbf{r} = \langle x,y,z \rangle$ and $\mathbf{r_0} = \langle x_0, y_0, z_0 \rangle$ in component form, we have

$$\langle a,b,c \rangle . \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

To obtain for the **scalar equation**, we can expand the dot product becomes

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

Theorem:
If a, b, c and d are constants and a, b and c are not all zero, then the equation of the plane can be rewritten as $ax+by+cz+d=0$ having the normal vector $\mathbf{n} = \langle a,b,c \rangle$.

3.5 EQUATIONS OF LINES AND PLANES

Example:
Find an equation of the plane passing through the point (3, -1, 7) and perpendicular to the vector $n = \langle 4, 2, -5 \rangle$

Solution:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$
$$4(x - 3) + 2(y + 1) - 5(z - 7) = 0$$
$$4x + 2y - 5z + 25 = 0$$

3.5 EQUATIONS OF LINES AND PLANES

Example

Find the equation of the plane passing through the points $P_1(1,2,-1)$, $P_2(2,3,1)$ and $P_3(3,-1,2)$.

Solution:

Since $P_1(1,2,-1)$, $P_2(2,3,1)$ and $P_3(3,-1,2)$ lies in the plane, the vectors $\overrightarrow{P_1P_2} = \langle 1,1,2 \rangle$ and $\overrightarrow{P_1P_3} = \langle 2,-3,3 \rangle$ are parallel to the plane.

Therefore $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \langle 9,1,-5 \rangle$ is normal to the plane.

($\because$ $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ is perpendicular to both $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$)

A point-normal form for the equation of a plane is (P_1 lies in the plane)

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

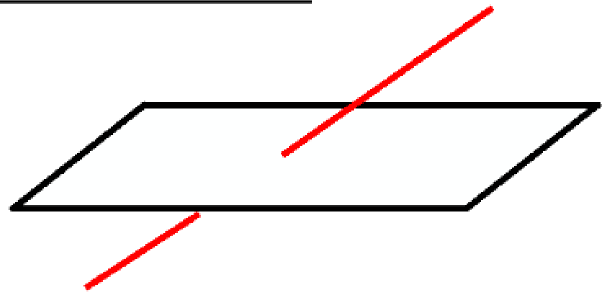
$$\Rightarrow 9(x-1) + 1(y-2) - 5(z+1) = 0 \Rightarrow 9x + y - 5z - 16 = 0$$

3.5 EQUATIONS OF LINES AND PLANES

The Intersection of a Line and a Plane

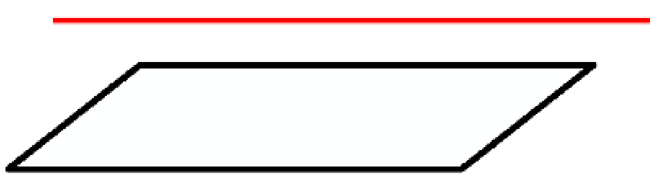
Types of Intersection:

1.) ***One Point of Intersection:***



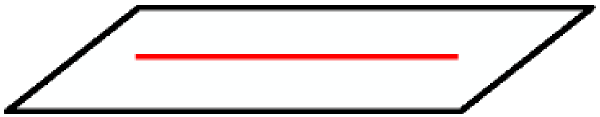
The line crosses the plane at a single point.

2.) ***No Point of Intersection:***



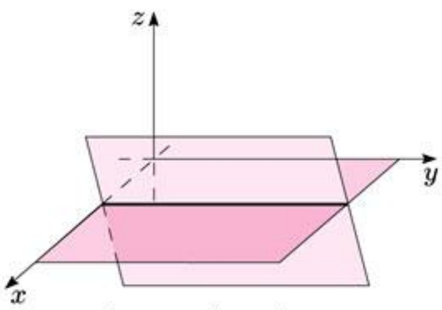
The line and plane are parallel and distinct.

3.) ***Infinite Points of Intersection:***

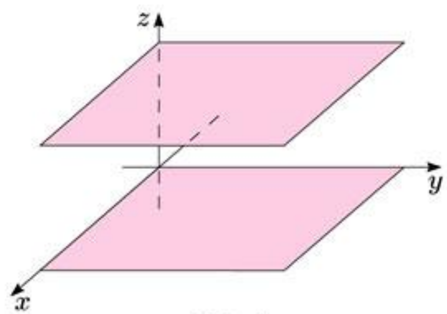


The line lies in the plane.

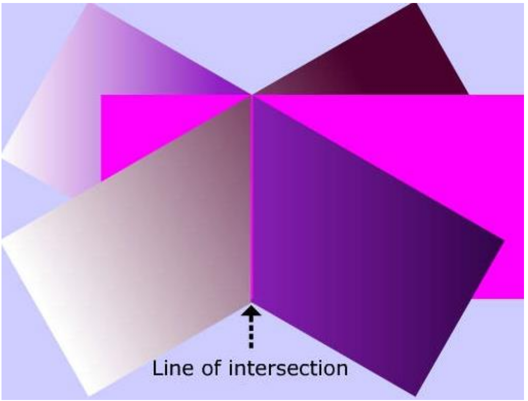
3.5 EQUATIONS OF LINES AND PLANES

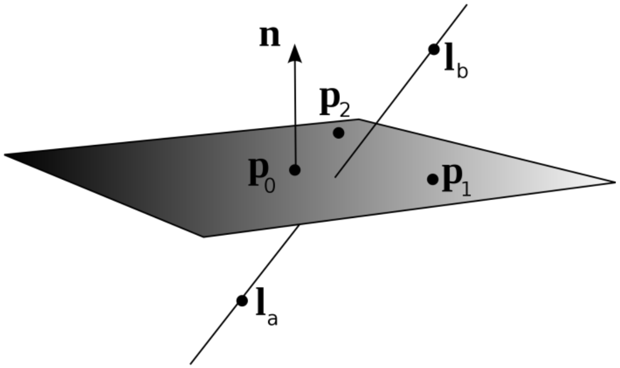


intersecting planes



parallel planes





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